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D-CONCURRENT VECTOR FIELD IN A FINSLER SPACE OF THREE-DIMENSIONS

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ABSTRACT

Concurrent vector fields in a Finsler space were first of all defined and studied by Tachibana [9] in 1950, followed by Matsumoto and Eguchi [2] and others. In 2004, Rastogi and Dwivedi [4] studied the existence of concurrent vector fields in a Finsler space of n-dimensions and showed that the definition of concurrent vector fields in its present form is unsuitable. Further, they modified the definition of concurrent vector fields in a Finsler space of n-dimensions. Recently, Rastogi [6], defined and studied weakly and partially concurrent vector fields in a Finsler space of three-dimensions. The purpose of the present paper is to define and study a vector field $X^i(x)$ *in* F^3*, called D-concurrent vector field, which is based on a tensor* D_{ijk} *defined and studied by Rastogi [5,7]. We have also defined and studied weakly and partially D-concurrent vector fields of various types and relationship between them.*

KEYWORDS: *Concurrent vector, Finsler Space, Three-Dimensions*

INTRODUCTION

In a three-dimensional Finsler space F^3, metric function is represented by L(x,y), metric tensor by $g_{ij} = l_i l_j + m_i m_j + n_i n_j$ and angular metric tensor by $h_{ij} = m_i m_j + n_i n_j$. The h- and v-covariant derivatives of unit vector fields l_i, m_i and n_i are given by [3], [8]:

$$l_{i/j} = 0,\ m_{i/j} = n_i h_j,\ n_{i/j} = -m_i h_j, \tag{1.1}$$

li//j = L-1 hij, mi//j = L-1(-li mj + ni vj), ni//j = -L-1(li nj + mi vj), (1.2)

where h_j and v_j are, respectively, h- and v-connection vectors in F^3. We have well-known torsion tensor C_{ijk} in F^3, defined as

$$C_{ijk} = C_{(1)} m_i m_j m_k - \sum_{(i,j,k)}\{C_{(2)} m_i m_j n_k - C_{(3)} m_i n_j n_k\} + C_{(2)} n_i n_j n_k \tag{1.3}$$

Rastogi and Dwivedi [4] have given the following modified definition of concurrent vector field in a Finsler space of n-dimensions.

Definition 1.: A vector field $X^i(x)$ in a Finsler space of n-dimensions F^n is said to be a concurrent vector field, if it satisfies (i) $X^i A_{ijk} = \alpha h_{jk}$ and (ii) $X^i_{/j} = -\delta^i_j$, where α is a non-zero arbitrary scalar function of x and y and other terms have their usual meaning.

Recently, Rastogi [5] has defined a third order symmetric tensor D_{ijk} in F^3, in the following form:

$$D_{ijk} = D_{(1)} m_i m_j m_k + D_{(2)} n_i n_j n_k + \sum_{(i,j,k)}\{D_{(3)} m_i m_j n_k + D_{(4)} m_i n_j n_k\}, \tag{1.4}$$

where $D_{ijk}\, l^i = 0$, $D_{ijk}\, g^{jk} = D_i = D\, n_i$, $D_{(2)} + D_{(3)} = D$ and $D_{(1)} + D_{(4)} = 0$.

Alternatively, D_{ijk} is expressed as

$$D_{ijk} = D_{(1)}\, m_i\, m_j\, m_k + D_{(2)}\, n_i\, n_j\, n_k + \sum_{(i,j,k)}\{D_{(3)}\, m_i\, m_j\, n_k - D_{(1)}\, m_i\, n_j\, n_k\} \tag{1.5}$$

Let us assume that there exists a vector field $X^i(x,y)$, in F^3 given by

$$X^i = \alpha\, l^i + \beta m^i + \gamma\, n^i. \tag{1.6}$$

Taking h-covariant derivative [3] of equation (1.6) and using (1.1), we get

$$X^i_{/r} = \alpha_{/r}\, l^i + (\beta_{/r} - \gamma\, h_r)\, m^i + (\gamma_{/r} + \beta\, h_r)\, n^i, \tag{1.7}$$

which by virtue of $X^i_{/r} = -\,\delta^i_r$, gives

$$\alpha_{/r} = -\, l_r,\ \underline{\beta}_{/r} = \gamma\, h_r - m_r,\ \gamma_{/r} = -\,(\beta\, h_r + n_r),\ \alpha_{/0} = -1,\ \beta_{/0} = \gamma\, h_0,\ \gamma_{/0} = -\beta\, h_0,$$

$$\alpha_{/r}\, m^r = 0,\ \beta_r\, m^r = \gamma\, h_r\, m^r - 1,\ \gamma_r\, m^r = -\,\beta\, h_r\, m^r,$$

$$\alpha_{/r}\, n^r = 0,\ \beta_r\, n^r = \gamma\, h_r\, n^r,\ \gamma_{/r}\, n^r = -\,(\beta\, h_r\, n^r + 1). \tag{1.8}$$

Further taking v-covariant derivative [3] of equation (1.6) and using (1.2), we get

$$X^i_{//r} = l^i\{\alpha_{//r} - L^{-1}(\beta\, m_r + \gamma\, n_r)\} + m^i\{\beta_{//r} + L^{-1}(\alpha\, m_r - \gamma v_r)\}$$

$$+ n^i\{\gamma_{//r} + L^{-1}(\alpha\, m_r + \beta\, v_r)\} \tag{1.9}$$

D-Concurrent Vector Field

Def. 2.1.: A vector field $X^i(x)$ shall be called a D-concurrent vector field, in a Finsler space of three-dimensions F^3, if it satisfies

$$\text{(i)}\ X^i_{/j} = -\,\delta^i_j,\ \text{(ii)}\ X^i\, D_{ijk} = \Theta(x,y)\, h_{jk}, \tag{2.1}$$

where $\Theta(x, y)$ is a non-zero scalar function of x and y.

Equations (1.5) and (2.1) by virtue of (1.6) shall give

$$\Theta\, h_{jk} = m_j\, m_k(\beta\, D_{(1)} + \gamma\, D_{(3)}) + n_j\, n_k(\gamma\, D_{(2)} - \beta\, D_{(1)})$$

$$+ (m_j\, n_k + m_k\, n_j)(\beta\, D_{(3)} - \gamma\, D_{(1)}) \tag{2.2}$$

Multiplying equation (2.2), respectively, by m^j and n^j, we get

$$\Theta\, m_k = m_k\,(\beta\, D_{(1)} + \gamma\, D_{(3)}) + n_k(\beta\, D_{(3)} - \gamma\, D_{(1)}) \tag{2.3 a}$$

and

$$\Theta\, n_k = n_k\,(\gamma\, D_{(2)} - \beta\, D_{(1)}) + m_k\,(\beta\, D_{(3)} - \gamma\, D_{(1)}) \tag{2.3 b}$$

From equations (2.3) a, b, we can obtain

$\Theta = \beta\, D_{(1)} + \gamma\, D_{(3)} = \gamma\, D_{(2)} - \beta\, D_{(1)}$, which means

$$2\beta\, D_{(1)} = \gamma(D_{(2)} - D_{(3)}),\ \beta\, D_{(3)} = \gamma\, D_{(1)} \tag{2.4}$$

From the equations given in (2.4), we easily get

$2\,D_{(1)}{}^2 - D_{(2)}\,D_{(3)} + D_{(3)}{}^2 = 0$ (2.5)

Hence

Theorem 2.1.: In a three-dimensional Finsler space F^3, if a vector field X^i is D-concurrent, coefficients $D_{(1)}$, $D_{(2)}$ and $D_{(3)}$ are related by equation (2.5).

Taking h-covariant derivative of equation (1.5), we obtain by virtue of equation (1.1)

$D_{ijk/r} = \{D_{(1)/r} - 3\,D_{(3)}\,h_r\}\,m_i\,m_j\,m_k + \{D_{(2)/r} - 3\,D_{(1)}\,h_r\}\,n_i\,n_j\,n_k$

$+\sum_{(I,j,k)}[\{D_{(3)/r} + 3\,D_{(1)}\,h_r\}\,m_i\,m_j\,n_k - \{D_{(1)/r} + (D_{(2)} - 2\,D_{(3)})h_r\}\,m_i\,n_j\,n_k]$, (2.6)

which for $Q_{ijk} = D_{ijk/0}$ gives [5]:

$Q_{ijk} = \{D_{(1)/0} - 3\,D_{(3)}\,h_0\}\,m_i\,m_j\,m_k + \{D_{(2)/0} - 3\,D_{(1)}\,h_0\}\,n_i\,n_j\,n_k + \sum_{(I,j,k)}[\{D_{(3)/0}$

$+ 3\,D_{(1)}\,h_0\}\,m_i\,m_j\,n_k - \{D_{(1)/0} + (D_{(2)} - 2\,D_{(3)})h_0\}\,m_i\,n_j\,n_k]$ (2.7)

If we take h-covariant derivative of equation (2.1) ii, we get on simplification by virtue of equations (1.7) and (1.8), following relation

$m_j\,m_k[\beta(D_{(1)/r} - 3\,D_{(3)}\,h_r) + \gamma(D_{(3)/r} + 3\,D_{(1)}\,h_r) - D_{(1)}\,m_r - D_{(3)}\,n_r - \Theta_{/r}]$

$+ n_j\,n_k[\gamma(D_{(2)/r} - 3\,D_{(1)}\,h_r) - \beta\{D_{(1)/r} + (D_{(2)} - 2\,D_{(3)})h_r\} + D_{(1)}\,m_r - D_{(2)}\,n_r - \Theta_{/r}]$

$+ (m_j\,n_k + m_k\,n_j)[\beta(D_{(3)/r} + 3\,D_{(1)}\,h_r) - \gamma\{D_{(1)/r} + (D_{(2)} - 2\,D_{(3)})h_r\}$

$- D_{(3)}\,m_r + D_{(1)}\,n_r] = 0$ (2.8)

Multiplying equation (2.8) by g^{jk}, we can obtain $2\,\Theta_{/r} = \gamma\,D_{/r} - D\,(\beta\,h_r + n_r)$, where we have used $D_{(2)} + D_{(3)} = D$. This with the help of equation (1.8) gives $\Theta_{/r} = (1/2)\,(\gamma\,D)_{/r}$. Hence:

Theorem 2.2.: In a three-dimensional Finsler space F^3, if a vector field X^i is D-concurrent, it satisfies $X^iQ_{ijk} = (1/2)\,(\gamma\,D)_{/0}\,h_{jk}$.

If we multiply equation (2.8), respectively, by m^j and n^j, we get

$\Theta_{/r} = \beta(D_{(1)/r} - 3\,D_{(3)}\,h_r) + \gamma(D_{(3)/r} + 3\,D_{(1)}\,h_r) - D_{(1)}\,m_r - D_{(3)}\,n_r$, (2.9)a

$\Theta_{/r} = \gamma(D_{(2)/r} - 3\,D_{(1)}\,h_r) - \beta\{D_{(1)/r} + (D_{(2)} - 2\,D_{(3)})h_r\} + D_{(1)}\,m_r - D_{(2)}\,n_r$ (2.9) b

and

$\beta(D_{(3)/r} + 3\,D_{(1)}\,h_r) - \gamma\{D_{(1)/r} + (D_{(2)} - 2\,D_{(3)})h_r\} - D_{(3)}\,m_r + D_{(1)}\,n_r] = 0$, (2.9) c

which when multiplied, respectively, by m^r and n^r give

$\Theta_{/r}\,m^r = \beta(D_{(1)/r}\,m^r - 3\,D_{(3)}\,h_{2)32}) + \gamma(D_{(3)/r}\,m^r + 3\,D_{(1)}\,h_{2)32}) - D_{(1)}$, (2.10) a

$\Theta_{/r}\,m^r = \gamma(D_{(2)/r}\,m^r - 3\,D_{(1)}\,h_{2)32}) - \beta\{D_{(1)/r}\,m^r + (D_{(2)} - 2\,D_{(3)})h_{2)32}\} + D_{(1)}$(2.10)b

$\Theta_{/r}\,n^r = \beta(D_{(1)/r}\,n^r - 3\,D_{(3)}\,h_{2)33}) + \gamma(D_{(3)/r}\,n^r + 3\,D_{(1)}\,h_{2)33}) - D_{(3)}$ (2.10) c

$\Theta_{/r}\, n^r = \gamma(D_{(2)/r}\, n^r - 3\, D_{(1)}\, h_{2)33}) - \beta\{D_{(1)/r}\, n^r + (D_{(2)} - 2\, D_{(3)})h_{2)33}\} - D_{(2)}$ (2.10) d

and

$\beta(D_{(3)/r}\, m^r + 3\, D_{(1)}\, h_{2)32}) - \gamma\{D_{(1)/r}\, m^r + (D_{(2)} - 2\, D_{(3)})h_{2)32}\} - D_{(3)} = 0$ (2.10) e

$\beta(D_{(3)/r}\, n^r + 3\, D_{(1)}\, h_{2)33}) - \gamma\{D_{(1)/r}\, n^r + (D_{(2)} - 2\, D_{(3)})h_{2)33}\} + D_{(1)} = 0.$ (2.10) f

From equations (2.10) a and (2.10) b, we can obtain

(2 Θ/r – γ D/r) mr + β D h2)32 = 0 (2.11) a

Similarly, from (2.10) c and (2.10) d, we get

$(2\, \Theta_{/r} - \gamma\, D_{/r})\, n^r + D\, (\beta\, h_{2)33} + 1) = 0$ (2.11) b

Hence

Theorem 2.3.: In a three-dimensional Finsler space, in case of a D-concurrent vector field X^i,

(i) coefficients $D_{(1)}$, $D_{(2)}$ and $D_{(3)}$ satisfy equations (2.10) a, b, c, d, e, f

(ii) $\Theta_{/r}$ satisfies equations (2.11) a, b.

Taking v-covariant derivative of equation (1.5) and using results of equation (1.2), we get on simplification

$D_{ijk//r} = B_{(1)r}\, m_i\, m_j\, m_k + B_{(2)r}\, n_i\, n_j\, n_k + \sum_{(I,j,k)} B_{(3)r}\, m_i\, m_j\, n_k + B_{(4)r}\, m_i\, n_j\, n_k$

$-L^{-1}\sum_{(I,j,k)} [l_i\, D_{(1)}\{m_r(m_j\, m_k - n_j\, n_k) - n_r(m_j\, n_k + m_k\, n_j)\} + D_{(2)}\, n_r\, l_i\, n_j\, n_k$

$+ D_{(3)}\{m_r\, n_k(l_i\, m_j + l_j\, m_i) + m_k\, n_r\, l_i\, m_j\}]$ (2.12)

where

B(1)r = D(1)//r – 3 L-1 D(3) vr, B(2)r = D(2)//r – 3 L-1 D(1) vr, (2.13) a

B(3)r = D(3)//r + 3 L-1 D(1) vr, B(4)r = D(1)//r + L-1(2 D(3) – D(2))vr (2.13) b

Taking v-covariant derivative of equation (2.1) (ii), using equations (1.5), (1.9) and (2.12) and multiplying the resulting equation by g^{jk}, we get on simplification

$2\, \Theta_{//r} = D[\gamma_{//r} + L^{-1}\alpha\, (m_r - n_r)] + 2\beta\, D_{(1)//r} + \gamma\, D_{//r,}$ (2.14)

which leads to

2 Θ//0 = (γ D)//0 + 2 β D(1)//0 (2.15) a

2 Θ//r mr = {(γD)//r + 2 β D(1)//r)} mr + L-1 D α (2.15) b

2 Θ//r nr = {(γD)//r + 2 β D(1)//r)} nr – L-1 D α (2.15) c

Hence

Theorem 2.4: In a three-dimensional Finsler space, in case of a D-concurrent vector field X^i, $\Theta_{//r}$ is given by equation (2.14) and satisfies (2.15) a, b, c.

Similar, to a C-reducible Finsler space Matsumoto [1], the author [5] has defined, D-reducible Finsler space F^3 in which the tensor D_{ijk} satisfies:

$$D_{ijk} = (1/4) \sum_{(I,j,k)}\{h_{ij} D_k\}, \tag{2.16}$$

which by virtue of equation (2.1), shall give

$$\Theta\, h_{jk} = (D/4)[\beta(m_j n_k + m_k n_j) + \gamma(h_{jk} + 2 n_j n_k)]. \tag{2.17}$$

Multiplying equation (2.17) by g^{jk}, we get on simplification

Theorem 2.5.: In a three-dimensional D-reducible Finsler space F^3, for a D-concurrent vector field $X^i(x)$, $\Theta = (1/2)\gamma D$.

Taking h-covariant differentiation of $2\Theta = \gamma D$, we get $2\Theta_{/r} = \gamma_{/r} D + \gamma D_{/r}$, which when compared with equation $2\Theta_{/r} = \gamma D_{/r} - D_r - \beta D h_r$, gives

$$D(\gamma_{/r} + \beta h_r + n_r) = 0 \tag{2.18}$$

From equation (2.18), we can obtain

$$\gamma_{/r} m^r + \beta h_{2)32} = 0,\ \gamma_{/r} n^r + \beta h_{2)33} + 1 = 0 \tag{2.19}$$

Hence:

Theorem 2.6.: In a three-dimensional D-reducible Finsler space F^3, for a D-concurrent vector field $X^i(x)$, β and γ satisfy equations given by (2.19).

Taking v-covariant differentiation of $2\Theta = \gamma D$, we get $2\Theta_{//r} = \gamma_{//r} D + \gamma D_{//r}$, which when compared with equation (2.14) leads to

$$2\beta D_{(1)//r} + L^{-1} D \alpha (m_r - n_r) = 0, \tag{2.20}$$

From equation (2.20), we easily obtain

$$D_{(1)//0} = 0,\ D_{(1)//r} m^r + D_{(1)//r} n^r = 0 \tag{2.21}$$

Hence:

Theorem 2.7.: In a three-dimensional D-reducible Finsler space F^3, for a D-concurrent vector field $X^i(x)$, scalar D satisfies equation (2.21).

3. WEAKLY D-CONCURRENT VECTOR FIELDS.

From equation (1.5), by virtue of $D_{ijk} m^k = {}'D_{ij}$ and $D_{ijk} n^k = {}^*D_{ij}$, we can get

‘Dij = D(1)(mi mj – ni nj) + D(3)(mi nj + mj ni) (3.1)

and

$${}^*D_{ij} = D_{(2)} n_i n_j + D_{(3)} m_i m_j - D_{(1)}(m_i n_j + m_j n_i), \tag{3.2}$$

which are symmetric tensors in i and j and satisfy

$$D_{ijk} = {}'D_{ij} m_k + {}^*D_{ij} n_k \tag{3.3}$$

From equations (3.1) and (3.2), we can get

$'D_i = 'D_{ij}\, m^j = D_{(1)}\, m_i + D_{(3)}\, n_i,\ ''D_i = 'D_{ij}\, n^j = D_{(3)}\, m_i - D_{(1)}\, n_i$ (3.4) a

$*D_i = *D_{ij}\, m^j = ''D_i,\ **D_i = *D_{ij}\, n^j = D_{(2)}\, n_i - D_{(1)}\, m_i,$ (3.4) b

such that $'D_{ij} = 'D_i\, m_j + *D_i\, n_j$ and $*D_{ij} = ''D_i\, m_j + **D_i\, n_j$.

Now, we shall give the following definitions:

Def. 3.1.: A vector field $X^i(x)$ in a Finsler space F^3 shall be called weakly D-concurrent vector field of first kind if (i) $X^i_{/j} = -\delta^i_j$ and (ii) $X^i\, 'D_i = \varphi(x, y)$, where $\varphi(x, y)$ is a non-zero scalar function of x and y.

From equation (3.4), a and this definition, we can get

$\varphi(x,y) = \beta\, D_{(1)} + \gamma\, D_{(3)}$ (3.5)

and

$\varphi_{/j} = \beta_{/j}\, D_{(1)} + \beta\, D_{(1)/j} + \gamma_{/j}\, D_{(3)} + \gamma\, D_{(3)/j}$ (3.6)

Substituting the values of $\beta_{/j}$ and $\gamma_{/j}$ from equation (1.8) in (3.6), we get

$\varphi_{/j} = \beta(D_{(1)/j} - D_{(3)}\, h_j) + \gamma(D_{(3)/j} + D_{(1)}\, h_j) - (D_{(1)}\, m_j + D_{(3)}\, n_j)$ (3.7)

which gives

$\varphi_{/0} = \{\beta(D_{(1)/0} - D_{(3)}\, h_0) + \gamma(D_{(3)/0} + D_{(1)}\, h_0)\},$ (3.8) a

$\varphi_{/j}\, m^j = \beta(D_{(1)/j}\, m^j - D_{(3)}\, h_{2)32}) + \gamma(D_{(3)/}\, m^j + D_{(1)}\, h_{2)32}) - D_{(1)}$ (3.8) b

and

$\varphi_{/j}\, n^j = \beta(D_{(1)/j}\, n^j - D_{(3)}\, h_{2)33}) + \gamma(D_{(3)/j}\, n^j + D_{(1)}\, h_{2)33}) - D_{(3)}$ (3.8) c

Hence

Theorem 3.1.: In a three-dimensional Finsler space F^3, for a weekly D-concurrent vector field of first kind, scalar φ satisfies equations (3.8) a, b, c.

Def. 3.2.: A vector field $X^i(x)$ in a Finsler space F^3 shall be called weakly D-concurrent vector field of second kind if (i) $X^i_{/j} = -\delta^i_j$ and (ii) $X^i\, ''D_i = \psi(x, y)$, where $\psi(x, y)$ is a non-zero scalar function of x and y.

Substituting the value of $''D_i$ from equation (3.3) and using Def. 3.2., we get

$\psi(x,y) = \beta\, D_{(3)} - \gamma\, D_{(1)}$ (3.9) a

Differentiating equation (3.9) a and using equation (1.8), we get

$\psi_{/j} = \beta(D_{(3)/j} + D_{(1)}\, h_j) - \gamma(D_{(1)/j} - D_{(3)}\, h_j) - D_{(3)}\, m_j + D_{(1)}\, n_j$ (3.9) b

which leads to

ψ/0 = β(D(3)/0 + D(1) h0) - γ(D(1)/0 – D(3) h0), (3.10) a

ψ/j mj = β(D(3)/j mj+ D(1) h2)32) – γ(D(1)/j mj – D(3) h2)32) – D(3) (3.10) b

$\psi_{/j}\, n^j = \beta(D_{(3)/j}\, n^j + D_{(1)}\, h_{2)33}) - \gamma(D_{(1)/j}\, n^j - D_{(3)}\, h_{2)33}) + D_{(1)}$ (3.10) c

Hence

Theorem 3.2.: In a three-dimensional Finsler space F^3, for a weakly D- concurrent vector field of second kind, ψ satisfies equations (3.10) a, b, c.

Def. 3.3.: A vector field $X^i(x)$ in a Finsler space F^3 shall be called weakly D-concurrent vector field of third kind if (i) $X^i_{/j} = -\delta^i_j$ and (ii) $X^i\,{}^{**}D_i = \omega(x,y)$, where $\omega(x,y)$ is a non-zero scalar function of x and y.

Substituting the value of $^{**}D_i$ from equation (3.4) in Def. 3.3, we get

$\omega(x,y) = \gamma\, D_{(2)} - \beta\, D_{(1)}$ (3.11) a

Differentiating equation (3.11) a and using equation (1.8), we can obtain

$\omega_{/j} = \gamma(D_{(2)/j} - D_{(1)}\, h_j) - \beta(D_{(1)/j} + D_{(2)}\, h_j) + D_{(1)}\, m_j - D_{(2)}\, n_j$ (3.11) b

From equation (3.11) b, we can obtain

$\omega_{/0} = \gamma(D_{(2)/0} - D_{(1)}\, h_0) - \beta(D_{(1)/0} + D_{(2)}\, h_0)$ (3.12) a

$\omega_{/j}\, m^j = \gamma(D_{(2)/j}\, m^j - D_{(1)}\, h_{2)32}) - \beta(D_{(1)/j}\, m^j + D_{(2)}\, h_{2)32}) + D_{(1)}$ (3.12) b

$\omega_{/j}\, n^j = \gamma(D_{(2)/j}\, n^j - D_{(1)}\, h_{2)33}) - \beta(D_{(1)/j}\, n^j + D_{(2)}\, h_{2)33}) - D_{(2)}$ (3.12) c

Hence

Theorem 3.3.: In a three-dimensional Finsler space F^3, for a weakly D-concurrent vector field of third kind, ω satisfies equations (3.12) a, b, c.

Using the fact that X^i is a function of x alone, we can observe that $X^i_{//r} = X^p\, C^i_{pr}$, which by virtue of equation (1.3), on simplification shall give

$X^i_{//r} = \beta\{C_{(1)}\, m^i m_r - C_{(2)}(m^i n_r + n^i m_r) + C_{(3)}\, n^i n_r\}$

$+ \gamma\{C_{(2)}\, n^i n_r - C_{(2)}\, m^i m_r + C_{(3)}\,(m^i n_r + n^i m_r)\}$ (3.13)

Comparing equations (1.9) and (3.13), we can observe that

$\alpha_{//r} - L^{-1}(\beta\, m_r + \gamma\, n_r) = 0,$ (3.14) a

$\beta_{//r} + L^{-1}(\alpha\, m_r - \gamma\, v_r) = (\beta\, C_{(1)} - \gamma\, C_{(2)})\, m_r + (\gamma\, C_{(3)} - \beta\, C_{(2)})\, n_r$ (3.14) b

$\gamma_{//r} + L^{-1}(\alpha\, m_r + \beta\, v_r) = (\gamma\, C_{(3)} - \beta\, C_{(2)})\, m_r + (\beta\, C_{(3)} + \gamma\, C_{(2)})\, n_r$ (3.14) c

Equations (3.14) a, b, c also give us $\alpha_{//0} = 0$, $\alpha_{//r}\, m^r = L^{-1}\beta$, $\alpha_{//r}\, n^r = L^{-1}\gamma$, $\beta_{//0} = 0$, $\beta_{//r}\, m^r = \beta\, C_{(1)} - \gamma\, C_{(2)} - L^{-1}(\alpha - \gamma\, v_{2)32})$, $\beta_{//r}\, n^r = \gamma\, C_{(3)} - \beta\, C_{(2)} + L^{-1}\gamma\, v_{2)33}\gamma_{//0} = 0$, $\gamma_{//r}\, m^r = \gamma\, C_{(3)} - \beta\, C_{(2)} - L^{-1}(\alpha + \beta\, v_{2)32})$, $\gamma_{//r}\, n^r = \beta\, C_{(3)} + \gamma\, C_{(2)} - L^{-1}\beta\, v_{2)33}$

Taking v-covariant derivative of equation (3.5), we get on simplification

$\varphi_{//r} = m_r\{D_{(1)}(\beta\, C_{(1)} - \gamma\, C_{(2)} - L^{-1}\alpha) + D_{(3)}(\gamma\, C_{(3)} - \beta\, C_{(2)} - L^{-1}\alpha)\}$

$+ n_r\{D_{(1)}(\gamma\, C_{(3)} - \beta\, C_{(2)}) + D_{(3)}\,(\beta\, C_{(3)} + \gamma\, C_{(2)})\}$

$$+ \beta(D(1)//r - L\text{-}1\ D(3)\ vr) + \gamma(D(3)//r + L\text{-}1\ D(1)\ vr) \quad (3.15)$$

From equation (3.15), we can get

$$\varphi//0 = \beta\ D(1)//0 + \gamma\ D(3)//0, \quad (3.16)\ a$$

$$\varphi//r\ mr = \{D(1)(\beta\ C(1) - \gamma\ C(2) - L\text{-}1\alpha) + D(3)(\gamma\ C(3) - \beta\ C(2) - L\text{-}1\alpha)\}$$

$$+ \beta(D(1)//r\ mr - L\text{-}1\ D(3)\ v2)32) + \gamma(D(3)//r\ mr + L\text{-}1\ D(1)\ v2)32) \quad (3.16)\ b$$

$$\varphi//r\ nr = \{D(1)(\gamma\ C(3) - \beta\ C(2)) + D(3)\ (\beta\ C(3) + \gamma\ C(2))\}$$

$$+\beta(D(1)//r\ nr - L\text{-}1\ D(3)\ v2)33) + \gamma(D(3)//r\ nr + L\text{-}1\ D(1)\ v2)33) \quad (3.16)\ c$$

Hence

Theorem 3.4.: In a three-dimensional Finsler space F^3, for a weakly D- concurrent vector field of first kind, scalar φ satisfies equations (3.16) a, b, c.

Similarly, from equation (3.9) a, we can obtain

$$\psi//r = mr\{D(3)(\beta\ C(1) - \gamma\ C(2) - L\text{-}1\alpha) - D(1)(\gamma\ C(3) - \beta\ C(2) - L\text{-}1\alpha)\}$$

$$+ nr\{D(3)(\gamma\ C(3) - \beta\ C(2)) - D(1)(\beta\ C(3) + \gamma\ C(2))\}$$

$$+ \beta(D(3)//r + L\text{-}1\ D(1)\ vr) - \gamma(D(1)//r - L\text{-}1\ D(3)\ vr) \quad (3.17)$$

which implies

$$\psi//0 = \beta\ D(3)//0 - \gamma\ D(1)//0 \quad (3.18)\ a$$

$$\psi//rmr= \{D(3)(\beta\ C(1) - \gamma\ C(2) - L\text{-}1\alpha) - D(1)(\gamma\ C(3) - \beta\ C(2) - L\text{-}1\alpha)\}$$

$$+ \beta(D(3)//r\ mr + L\text{-}1\ D(1)\ v2)32) - \gamma(D(1)//r\ mr - L\text{-}1\ D(3)\ v2)32) \quad (3.18)\ b$$

$$\psi//r\ nr = \{D(3)(\gamma\ C(3) - \beta\ C(2)) - D(1)(\beta\ C(3) + \gamma\ C(2))\}$$

$$+ \beta(D(3)//r\ nr + L\text{-}1\ D(1)\ v2)33) - \gamma(D(1)//r\ nr - L\text{-}1\ D(3)\ v2)33) \quad (3.18)\ c$$

Hence

Theorem 3.5.: In a three-dimensional Finsler space F^3, for a weakly D-concurrent vector field of second kind, ψ satisfies equations (3.18) a, b, c.

From equation (3.11) a, we can obtain

$$\omega//r = mr\{D(2)(\gamma\ C(3) - \beta\ C(2) - L\text{-}1\alpha) - D(1)(\beta\ C(1) - \gamma\ C(2) - L\text{-}1\alpha)\}$$

$$+ nr\{D(2)(\beta\ C(3) + \gamma\ C(2)) - D(1)(\gamma\ C(3) - \beta\ C(2))\}$$

$$+ \gamma(D(2)//r - L\text{-}1\ D(1)\ vr) - \beta(D(1)//r + L\text{-}1\ D(2)\ vr) \quad (3.19)$$

which leads to

$$\omega_{//0} = \gamma\ D_{(2)//0} - \beta\ D_{(1)//0} \quad (3.20)\ a$$

$\omega_{//r}\, m^r = \{D_{(2)}(\gamma\, C_{(3)} - \beta\, C_{(2)} - L^{-1}\alpha) - D_{(1)}(\beta\, C_{(1)} - \gamma\, C_{(2)} - L^{-1}\alpha)\}$

$+ \gamma(D_{(2)//r}\, m^r - L^{-1}\, D_{(1)}\, v_{2)32}) - \beta(D_{(1)//r} + L^{-1}\, D_{(2)}\, v_{2)32})$ (3.20) b

$\omega//r\ nr = \{D(2)(\beta\, C(3) + \gamma\, C(2)) - D(1)(\gamma\, C(3) - \beta\, C(2))\}$

$+ \gamma(D_{(2)//r}\, n^r - L^{-1}\, D_{(1)}\, v_{2)33}) - \beta(D_{(1)//r}\, n^r + L^{-1}\, D_{(2)}\, v_{2)33})$ (3.20) c

Hence

Theorem 3.6.: In a three-dimensional Finsler space F^3, for a weakly D-concurrent vector field of third kind, ω satisfies equations (3.20) a, b, c.

In a D-reducible Finsler space F^3, equation (2.16), by virtue of (3.1) and (3.2) gives

$'D_{ij} = (D/4)\,(m_i\, n_j + m_j\, n_i),\ {}^*D_{ij} = (D/4)(m_i\, m_j + 3\, n_i\, n_j)$ (3.21) a

while using equations (3.4) a, b, we get

$'D_i = (D/4)\, n_i,\ {}^*D_i = {}''D_i = (D/4)\, m_i,\ {}^{**}D_i = 3(D/4)\, n_i$ (3.21) b

From these equations, we can obtain

$D_{(1)} = 0,\ D_{(2)} = 3D/4,\ D_{(3)} = D/4$ (3.21) c

and also

$X^i\ 'D_i = \gamma D/4,\ X^i\ {}^*D_i = \beta\, D/4,\ X^i\ {}^{**}D_i = 3\gamma D/4$ (3.22) a

$X^i\ 'D_{ij} = (D/4)(\gamma\, m_j + \beta\, n_j),\ X^i\ {}^*D_{ij} = (D/4)(\beta\, m_j + 3\,\gamma\, n_j)$ (3.22) b

Hence

Theorem 3.7.: In a three-dimensional D-reducible Finsler space F^3, coefficients $D_{(1)}$, $D_{(2)}$ and $D_{(3)}$ are given by (3.21) c, while weakly and partially D-concurrent vector fields, respectively, satisfy equations (3.22) a and (3.22) b.

4. PARTIALLY D-CONCURRENT VECTOR FIELD OF FIRST KIND.

Def. 4.1.: A vector field $X^i(x)$, in a three-dimensional Finsler space F^3, shall be called partially D-concurrent vector field of first kind, if it satisfies

(i) $X^i_{/j} = -\,\delta^i_j$, (ii) $X^i\ 'D_{ij} = \Theta_j(x,y)$, (4.1)

where $\Theta_j(x,y)$ is a non-zero vector function of x and y.

From equations (3.1) and (4.1), we can get

$\Theta_j = D_{(1)}(\beta\, m_j - \gamma\, n_j) + D_{(3)}(\gamma\, m_j + \beta\, n_j)$ (4.2)

With the help of equations (3.5) and (3.9) a, we can get

$\Theta_j = \varphi\, m_j + \psi\, n_j$ (4.3)

showing that

$\Theta j\ lj = 0$, $\Theta j\ mj = \varphi$ and $\Theta j\ nj = \psi$ (4.4) a

$\Theta j/k = (\varphi/k - \psi\ hk)\ mj + (\psi/k + \varphi\ hk)\ nj$ (4.4) b

$\Theta j/k\ lj = 0$, $\Theta j/k\ lk = (\varphi/0 - \psi\ h0)\ mj + (\psi/0 + \varphi\ h0)\ nj$ (4.4) c

$\Theta j/k\ mj = \varphi/k - \psi\ hk$, $\Theta j/k\ nj = \psi/k + \varphi\ hk$ (4.4) d

$\Theta j/k\ mk = (\varphi/k\ mk - \psi\ h2)32)\ mj + (\psi/k\ mk + \varphi\ h2)32)\ nj$ (4.4) e

$\Theta j/k\ nk = (\varphi/k\ nk - \psi\ h2)33)\ mj + (\psi/k\ nk + \varphi\ h2)33)\ nj$ (4.4) f

Hence

Equations (4.3) and (4.4) show that partially D-concurrent vector field of first kind is a combination of weakly D-concurrent vector fields of first and second kind.

Theorem 4.1.: The partially D-concurrent vector field of first kind implies the existence of weakly D-concurrent vector fields of first and second kind, but the converse is not true.

From equation (4.3), we can also obtain

$\Theta_{j//k} = (\varphi_{//k} - L^{-1}\ \psi\ v_k)\ m_j + (\psi_{//k} + L^{-1}\ \varphi\ v_k)\ n_j - L^{-1}\ l_j\ \Theta_k$, (4.5)

which leads to

$\Theta j//k\ lj = - L\text{-}1\ \Theta k$, $\varphi j//k\ lk = (\varphi//0\ mj + \psi//0\ nj)$ (4.6) a

$\Theta j//k\ mj = \varphi//k - L\text{-}1\ \psi\ vk$, $\Theta j//k\ nj = \psi//k + L\text{-}1\ \varphi\ vk$ (4.6) b

$\Theta_{j//k}\ m^k = (\varphi_{//k}\ m^k - L^{-1}\ \psi\ v_{2)32)})\ m_j + (\psi_{//k}\ m^k + L^{-1}\ \varphi\ v_{2)32)})\ n_j - L^{-1}\ l_j\varphi$ (4.6) c

$\Theta_{j//k}\ n^k = (\varphi_{//k}\ n^k - L^{-1}\ \psi\ v_{2)33)})\ m_j + (\psi_{//k}\ n^k + L^{-1}\ \varphi\ v_{2)33)})\ n_j - L^{-1}\ l_j\ \psi$ (4.6) d

Hence

Theorem 4.2.: In a Finsler space F^3, for D-partially concurrent vector field of first kind, vector field Θ_j satisfies equations (4.5), (4.6) a, b, c, d.

Def. 4.2.: A vector field $X^i(x)$ in a Finsler space F^3, shall be called D-partially concurrent vector field of second kind, if it satisfies (i) $X^i_{/j} = - \delta^i_j$, (ii) $X^i\ {}^*D_{ij} = \varphi_j(x,y)$, (4.7)

where $\varphi_j(x,y)$ is a non-zero vector function of x and y.

From equations (3.2) and (4.7), we can get

$\varphi j = (\beta\ D(3) - \gamma\ D(1))\ mj + (\gamma\ D(2) - \beta\ D(1))\ nj$ (4.8)

which by virtue of (3.9) a and (3.11) a, leads to

$\varphi_j = \psi\ m_j + \omega\ n_j$ (4.9)

Hence

Theorem 4.3.: The partially D-concurrent vector field of second kind implies the existence of weakly D-concurrent vector fields of second and third kind, but the converse is not true.

From equation (4.9), by taking h-covariant derivative, we can easily obtain

φj/k = (ψ/k – ω hk) mj + (ω/k – ψ hk) nj (4.10) a

φj/k lj = 0, φj/k lk = (ψ/0–ω h0) mj + (ω/0 – ψ h0) nj (4.10) b

φj/k mj = ψ/k – ω hk, φj/k nj = ω/k – ψ hk (4.10) c

φj/k mk = (ψ/k mk – ω h2)32) mj + (ω/k mk – ψ h2)32) nj (4.10) d

φj/k nk = (ψ/k nk– ω h2)33) mj + (ω/k nk – ψ h2)33) nj (4.10) e

Hence

Theorem 4.4.: In a Finsler space, F^3, for D-partially concurrent vector field of second kind, vector field φ_j satisfies equations (4.10) a, b, c, d, e.

If we take v-covariant derivative, equation (4.9) will lead to

$\varphi_{j//k} = (\psi_{//k} – L^{-1}\omega\, v_k)\, m_j + (\omega_{//k} + L^{-1}\psi\, v_k)\, n_j – L^{-1}\, l_j\varphi_k$ (4.11) a

which implies

φj//k lj = - L-1φk,φj//k lk = ψ//0 mj + ω//0 nj (4.11) b

φj//k mj = ψ//k – L-1ω vk, φj//k nj = ω//k + L-1ψ vk (4.11) c

φj//k mk =(ψ//k mk– L-1ω v2)32) mj + (ω//k mk + L-1ψ v2)32) nj – L-1 ljψ (4.11) d

φj//k nk = (ψ//k nk – L-1ω v2)33) mj + (ω//k nk + L-1ψ v2)33) nj – L-1 ljω (4.11) e

Hence

Theorem 4.5.: In a Finsler space F^3, for D-partially concurrent vector field of second kind, vector field $\varphi_{j//k}$ satisfies equations (4.11) a, b, c, d, e.

Remark. If we observe equations (1.4), (3.1) and (3.2), we can notice that tensor D_{ijk} = ‘D_{ij} m_k + *D_{ij} n_k; therefore, it is obvious that D-concurrent vector field is a combination of D-partially concurrent vector fields of first and second kind, but the converse is not true.

5. CURVATURE PROPERTIES.

If D'_{ijkr} is a tensor based on D_{ijk} and defined as Rastogi [4]:

D’ijkr = Dirp Dpjk – Dikp Dpjr (5.1)

we can easily obtain from Def. 2.1

$X^i D'_{ijkr} = \Theta\, (h_{rp}\, D^p_{jk} – h_{kp}\, D^p_{jr}) = 0.$ (5.2)

Hence

Theorem 5.1.: In a three-dimensional Finsler space F^3, the curvature tensor D'_{ijkr} with a D-concurrent vector field $X^i(x)$satisfies equation (5.2).

In an earlier paper [7], I have defined a curvature tensor U_{ijkh} as follows:

$$U_{ijkh} = Ç_{(I,j)}\{D_{jkh/I} + D_{ikr}\, Q^r_{jh}\}, \quad (5.3)$$

Multiplying equation (5.3) by X^i, we get on simplification

$$X^i\, U_{ijkh} = X^i\, D_{jkh/I} – D_{jkh} – Ө_{/j}\, (m_k\, m_h + n_k\, n_h),$$

which can be expressed as

Xi Uijkh = (Xi 1Ai – D(1)) mj mk mh+ (Xi3Ai – D(2)) nj nk nh

+ (Xi4Ai – D(3))∑(j,k,h) mj mk nh – (Xi2Ai – D(1))∑(j,k,h) mj nk nh

-Ө/j (mk mh + nk nh), (5.4)

where

1Ai = D(1)/I – 3 D(3) hi, 2Ai = D(1)/I + (D(2) – 2 D(3))hi, (5.5) a

3Ai = D(2)/I – 3 D(1) hi, 4Ai = D(3)/I + 3 D(1) hi (5.5) b

1A0 = 1Ai li, 2A0 = 2Ai li, 3A0 = 3Ai li, 4A0 = 4Ai li (5.5) c

It is known that the tensor U_{ijkh} can also be expressed as Rastogi [7]:

Uijkh = 1Aij mk mh + 2Aij mk nh + 3Aij nk mh + 4Aij nk nh, (5.6)

where

1Aij = Ç(I,j)[1Ai mj + 4Ai nj + {D(3)(1A0 – 2A0) – 2 D(1)4A0} mj ni] (5.7) a

2Aij = Ç(I,j)[4Ai mj – 2Ai nj + {D(3)(4A0 – 3A0) + 2 D(1)2A0} mj ni] (5.7) b

3Aij = Ç(I,j)[4Ai mj – 2Ai nj + {D(1)(2A0 – 1A0) + (D(2) – D(3)) 4A0} mj ni] (5.7) c

4Aij = Ç(I,j)[2Aj mi – 3Aj ni + {D(1)(3A0 – 4A0) – (D(2) – D(3))2A0} mj ni]; (5.7) d

therefore, it is also possible to write equation (5.4) in an alternative form

$X^i\, U_{ijkh} = B_{(1)}\, m_j\, m_k\, m_h + B_{(2)}\, n_j\, n_k\, n_h + B_{(3)}\, m_j\, m_k\, n_h$

+ B(4)(mk mh nj + mh mj nk) – B(5) (mj nk nh + mk nh nj)

- B(6) mh nj nk – 2 Ө/j (mk mh + nk nh), (5.8)

where

B(1) = Xi1Ai + Ө 1A0 – D(1) + β(1A0 D(1) + 4A0 D(3)) + γ(4A0 D(1) – 2A0 D(3)),

B(2) = Xi 3Ai + Ө3A0 – D(2) – β(4A0 D(1) + 2A0 D(2)) + γ(3A0 D(2) + 2A0 D(1)),

B(3) = Xi4Ai + Ө4A0 – D(3) +β(4A0 D(1) – 2A0 D(3)) + γ(3A0 D(3) – 2A0 D(1)), B(4) = Xi4Ai + Θ4A0 + β(1A0 D(3) – 4A0 D(1)) + γ(4A0 D(3) + 2A0 D(1)),

B(5) = Xi2Ai + Θ 2A0 – D(1) – β(4A0 D(3) + 2A0 D(1)) + γ(3A0 D(1) +2A0 D(3)),

B(6) = Xi 2Ai + Θ 2A0 + D(3) – D(1) + β(1A0 D(1) – 4A0 D(2)) + γ(4A0 D(1) + 2A0 D(2))

Hence

Theorem 5.2.: In a three-dimensional Finsler space F^3, the curvature tensor U_{ijkh}, with a D-concurrent vector field $X^i(x)$ satisfies equation (5.4) or (5.8).

The author [7] has defined a tensor V_{ijkh}, in F^3, as follows:

$$V_{ijkh} = L\, D_{ijk//h} + l_h D_{ijk} + l_k D_{ijh} + l_j D_{ikh} + l_i D_{jkh} \tag{5.9}$$

From equation (5.9), on simplification by virtue of equations (1.5), (1.6), (2.1) and (2.6), we can obtain

$$X^i V_{ijkh} = L\, [\beta\{{}^1A_h\, m_j\, m_k - {}^2A_h\, n_j\, n_k + {}^4A_h(m_j\, n_k + m_k\, n_j)\}$$

$$+ \gamma\{{}^4A_h\, m_j\, m_k + {}^3A_h\, n_j\, n_k - {}^2A_h(m_j\, n_k + m_k\, n_j)\}]$$

$$+ \Theta(lj\ hkh + lk\ hjh + lh\ hjk) + \alpha\ Djkh \tag{5.10}$$

which implies

Theorem 5.3.: In a three-dimensional Finsler space F^3, the curvature tensor V_{ijkh}, with a D-concurrent vector field $X^i(x)$ satisfies equation (5.10).

REFERENCES

1. *Matsumoto, M: C-reducible Finsler spaces, Tensor, N. S., 24(1972), 29–37.*
2. *Matsumoto, M. and Eguchi, H.: Finsler spaces admitting concurrent vector field, Tensor, N. S., 28(1974), 239–249.*
3. *Matsumoto, M.: Foundations of Finsler Geometry and special Finsler spaces, Kaiseisha Press, Otsu, Japan, 1986.*
4. *Rastogi, S. C. and Dwivedi, A. K.: On the existence of concurrent vector fields in a Finsler space, Tensor, N. S., 65(2004), 48–54.*
5. *Rastogi, S. C.: On some new tensors and their properties in a Finsler space-I, Int. J. Res. Eng. Technol. , 7, issue 4(2019), 9–20.*
6. *Rastogi, S. C.: Weakly and Partially Concurrent vector fields in a Finsler space of three-dimensions, Int. J. Adv. Innov. Res., Vol. 8, issue 6, (2019), 1–9.*
7. *Rastogi, S. C.: Study of certain new tensors in a Finsler space of three-dimensions, Int. J. Res. Appl. Nat. Soc. Sci., 7, issue 11, (2019), 1–14.*
8. *Rund, H.: The differential geometry of Finslr spaces, Springer-Verlag, 1959.*
9. *Tachibana, S.: On Finsler spaces which admit concurrent vector field, Tensor, N. S., 1(1950), 1–5.*

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